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Some Circular Curves Generated by Pencils of Stelloids and their Polars

SOME CIRCULAR CURVES GENERATED BY PENCILS OF STELLOIDS AND THEIR POLARS

BY

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M. S. University of Illinois, 1914.

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

DOCTOR OF PHILOSOPHY

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1917

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

May 5 1917

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY Clarence Mark Hebbert

ENTITLED Some Circular Curves Generated by Pencils of Stelloids and their Polars

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE OF Doctor of Philosophy

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SOME CIRCULAR CURVES GENERATED BY PENCILS
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I. INTRODUCTION.

If $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ is a polynomial in $z = x + iy$, the reals may be separated from the imaginaries, and the expression put in the form $f(z) = u + iv$, where u and v are polynomials of the n^{th} degree in x and y with real coefficients. The curves defined by the equations $u = 0$ and $v = 0$ are called stelloids by Lucas,[1] rhizic curves by Walton,[2] potential curves by Kasner,[3] and orthic curves by Brooks.[4] The name given by Lucas is based upon the fact that the asymptotes are concurrent and divide the whole angle about their common point into n equal parts. Kasner calls them potential curves because they satisfy Laplace's differential equation and arise in connection with the study of potential in physical problems.[5] Much of Lucas' work

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[1] F. Lucas. Géométrie des polynômes, Journal de l'École Polytechnique, XLVI^e Cahier (1879), pp. 1-33. See page six of his article for the name stelloid and his reasons for using it.

[2] W. Walton. Several papers in Quarterly Journal of Mathematics XI (1871).

[3] E. Kasner. On the Algebraic Potential Curves, Bulletin of American Mathematical Society, VII (1901), p. 392.

[4] C. E. Brooks. A Note on the Orthic Cubic Curve, Johns Hopkins University Circular (1904$_2$) 47-52. Orthic Curves; or Algebraic curves which satisfy Laplace's equation in two dimensions. Proc. Amer. Phil. Soc. 43 (1904) 294-331.

[5] E. Picard, Revue Annuelle d'Analyse, Rendiconti del Circolo Matematico di Palermo, V (1891), p. 80. Reprinted from Reve Generale des Sciences Pures et Appliquées I (1890).

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was done from the latter standpoint.[1] From a purely geometric view-point the name stelloid seems most appropriate for these curves, and it will be used in this paper.

The properties of stelloids have been set forth by the men mentioned above and by Fouret.[2] Lucas showed (l.c.,p.8) that the polar of a point with respect to a stelloid is also a stelloid. Kasner (l.c.) and Emch[3] proved this by different methods.

Emch has shown (l.c.) that a pencil of stelloids of order $n + 1$ is completely determined by $n + 1$ distinct real points (A) and their associates.[4] The first polars of an arbitrary point (x', y') with respect to this pencil form a pencil of stelloids of order n, whose n^2 base-points (n real points and their associates) define with (x', y') a $(1, n^2)$ correspondence in the Cartesian plane. The real part $(1, n)$ of this correspondence is realized in a super-

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1 F. Lucas. Statique des Polynômes, Bulletin de Societe Mathématique de France XVII (1889), p. 17. Determination électrique des racines réeles et imaginaires de la derivée d'un polynôme quelconque. Comptes Rendus, CVI (1888), p. 195.

2 G. Fouret. Sur quelques proprietes géometriques des stelloides. Comptes Rendus CVI (1888), p. 342-, p. 966.

3 A. Emch. On Conformal Rational Transformations in a Plane. Rendiconti del Circolo Matematico di Palermo XXXIV (1912), pp.1-12. For other references on stelloids see G. Loria, Spezielle Algebraische und Transcendente Ebene Kurven, 15th Kap. - Geometrie der Polynome, Vol. I (1902), pp. 368-80.

4 G. Darboux defines associate points in Sur une classe remar quable de courbes et de surfaces algébriques, p. 61. See also E. Study, Vorlesungen über ausgewählte Gegenstände der Geometrie, I. Heft, pp. 8-19, Loria (l.c. above) p. 377, and Briot et Bouquet, Théorie des fonctions elliptiques (book IV, Ch. II). A. Cayley, Collected Works, Vol. VI, p. 499, calls them anti-points.

posed complex plane by the transformation

$$z' = z - \frac{(n+1)\ f(z)}{f'(z)},$$

in which the roots of $f(z) = 0$ represent the points (A) and $f'(Z)$ is the derivative of $f(z)$. The product of a pencil of stelloids of order $n + 1$ and the first polar pencil of a point (x', y') with respect to this pencil is a circular curve of order $2n + 1$ having an n-fold point at each of the circular points. This may be extended to the r^{th} polar pencil since the polynomials u, v, $\Delta^r u$, $\Delta^r v$, satisfy the conditions of the general theorem on p. 4 of the Rendiconti article referred to above, viz., "if u, v, and r, s are two pairs of irreducible polynomials in x and y with real coefficients of degree <u>m</u> and <u>n</u> respectively, (and satisfy Laplace's differential equation) each of the curves $ru + sv = 0$ and $rv - su = 0$ contains the circular points as multiple points; the degree of multiplicity is identical with that of the polynomial of lowest degree among u, v, r, s." Then we can state the

Theorem I. <u>The product of the pencil of stelloids of order $n + 1$ and the pencil of r^{th} polars (of order $n - r + 1$) is a curve of order $2n - r + 2$, having an $(n - r + 1)$-fold point at each of the circular points</u>.

The following table shows the curves generated in this manner.

n	1st polar	2nd polar	3rd polar	Product of curve and 1st polar	2nd polar	3rd polar	Product of 1st polar and 2nd polar	3rd polar	4th polar	
2	1			3circ						
3	2	1		5 bicirc	4circ		3 circular			
4	3	2	1	7 tricirc	6 bicirc	5circ	5 bicirc	4 circ		
5	4	3	2	9 4-circ	8 tricirc	7 bicirc	7 tricirc	6 bicirc	5circ	
⋮	⋮	⋮	⋮							
K	K-1	K-2	K-3							

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The table, in which the numbers in the columns indicate the orders of the curves, shows at once that it is impossible to generate in this way a bicircular quartic, tricircular sextic, or, in general, an n-ic having an $\frac{n}{2}$-fold point at each of the circular points. This is also evident from the fact that the products are of order m + n, generated by two pencils of curves of orders m and n, respectively, where $n \geqq m + 1$, and the product has an m-fold point at each of the circular points.

Emch has also made a detailed study of the stelloids connected with the circular transformation $z' = \frac{az + b}{cz + d}$, and, in particular $z' = \frac{1}{z}$. This is an involutoric circular transformation and is shown to be identical with a Steinerian transformation,[1] based upon a pencil of equilateral hyperbolas. The pencil of stelloids in this case consists of the equilateral hyperbolas through the points +1 and -1 and their associates. The first polar pencil is a pencil of straight lines and the product of the two pencils is a circular cubic, an invariant cubic in the Steinerian transformation.

It is the purpose of this paper to consider the transformation $z' = \frac{1}{z^2}$, which has the three cube roots of unity for double points and with which is connected the pencil of stelloids (cubics) through the three cube roots of unity and their associates. Some

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[1] A. Emch. Involutoric Circular Transformations as a Particular Case of the Steinerian Transformation and their Invariant nets of Cubics. Annals of Mathematics, 2nd Series, XIV (1912) pp. 57-71.

[illegible]

properties of the quintic generated by the pencil of cubics and the first polar pencil (equilateral hyperbolas) will be derived.

The transformation $z' = \frac{1}{z^n}$ will also be studied and the general form of the product of the pencil of stelloids through the $n + 1^{th}$ roots of unity and their associates, and the first and second polar pencil of any point (x',y') will be determined. Some properties of the asymptotes and foci of these curves will be derived. This transformation is simply the contracted form of the general transformation $z' = z - \frac{(n+1)f(z)}{f'(z)}$, for $f(z) = z^{n+1} - 1$.

For some of the work, use will be made of the following

Theorem II. <u>The product of a pencil of curves and the second polar pencil of a point (x',y') is identical with the polar of the product of the pencil and the first polar pencil of (x',y').</u>[1]

For, let the pencil of curves be

(1) $P + \lambda Q = 0$, and the first and second polar pencils,

(2) $\Delta P + \lambda \Delta Q = 0$, and

(3) $\Delta^2 P + \lambda \Delta^2 Q = 0$, respectively,

The product of pencils (1) and (2) is

(4) $P \cdot \Delta Q - Q \cdot \Delta P = 0$, whose polar is

$\Delta P \cdot \Delta Q + P \cdot \Delta^2 Q - \Delta Q \cdot \Delta P - Q \cdot \Delta^2 P = 0$ or

(5) $P \cdot \Delta^2 Q - Q \cdot \Delta^2 P = 0$, which is identical with the

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[1] It is understood that the pencils are projective.

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product[1] of (1) and (3).

II. Transformation $z' = z - \frac{3(z^3 - 1)}{3z^2} = \frac{1}{z^2}$

Geometrically, this transformation represents an inversion, a reflexion, doubling of the angle and squaring of the absolute value. For it may be replaced by the two transformations, $z'' = \frac{1}{z}$ and $z' = z''^2$, whose properties are well known. Straight lines are reflected on the x-axis and their inclinations are doubled. The unit circle corresponds to itself but only the three points (1,0), $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$, and $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ are invariant. The three lines joining these three points and the origin are also invariant lines but not point-wise. An equilateral hyperbola, $xy = c$, goes into the circle, $2c(x^2 + y^2) + y = 0$, counted twice. If (x', y') describes a straight line, the point (x, y) describes a locus of the fourth order, since the points corresponding to (x', y')* are the base-points of the first polar pencil of (x', y') with respect to the pencil of cubics (stelloids) through the three cube roots of unity and their associates.** Since the two transformations $z'' = \frac{1}{z}$ and $z' = z''^2$ are con-

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[1] On products of projective pencils see:
Clebsch, Vorlesungen über Geometrie, Vol. I (1876) p. 375.
Cremona, Theorie der Ebenen Kurven (German by Curtze, 1865) Paragraph 50, ff..
Sturm, Die Lehre von den Geometrischen Verwandtschaften, Vol. I (1909), p. 249, ff..
Ency. der Math. Wiss. III 2, 3 p. 353, ff.

* L. Cremona. Theorie der Ebenen Kurven (German by Curtze, 1865) p. 120, Lehrsatz XI.

** A. Emch. (l.c. p. 2.) pp. 8 and 12.

formal around all points except 0 and ∞, the result of using both of them is conformal, i.e., finite singularities of curves are preserved in the transformation $z' = \frac{1}{z^2}$. Infinite points, however, are transformed into singularities at the origin.

The pencil of cubics is $u + \lambda v = 0$ where $\underline{u}$ and $\underline{v}$ are the real and imaginary parts, respectively, of $z^3 - 1$, i.e.,

(1) $$u + \lambda v = x^3 - 3xy^2 - 1 + \lambda(3x^2y - y^3) = 0.$$

The projective pencil of first polars is

(2) $$(x^2 - y^2)x' - 2xyy' - 1 + \lambda[2xyx' + (x^2-y^2)y'] = 0,$$

pencil of second polars is

(3) $$(x'^2 - y'^2)x - 2x'y'y - 1 + \lambda[2x'y'x + (x'^2-y'^2)y] = 0.$$

The product of (1) and (2) is, as we should expect from the general theory, a bicircular quintic

(4) $$(x'y - y'x)[(x^2+y^2)^2 + 2x] + (y' - y)(3x^2-y^2) = 0.$$

The product of (1) and (3) is

(5) $$2(x^2+y^2)(xx'+yy')(x'y-xy')+(x'^2-y'^2)y+2xx'y'$$
$$-3x^2y' - 3x^2y + y^3 = 0,$$

a circular quartic; the first polar of $(x'y')$ with respect to (4), in agreement with Theorem II.

The product of (2) and (3) is the circular cubic

(6) $$(x'^2-y'^2)[(x'y+xy')(x^2+y^2)+y]$$
$$+2x'y'(x^2yy'+y'y^3-xx'y^2-x'x^3+x)-2xx'y-y'(x^2-y^2) = 0.$$

This cubic belongs to the class discussed by Emch in the paper referred to on p. 4, and will not be studied here.

<u>The Quintic</u> (4)

Since there are no terms of the fourth degree in equation (4),

and $(x'y - y'x)$ is a factor of the fifth degree terms, the line $x'y - y'x = 0$ is an asymptote. There is a double point at the origin and at each of the circular points, I and J, at infinity. The curve passes through the base-points of (1) and (2), viz., the points

$(1,0)$; $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$; $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$; and $\left(\pm\frac{1}{2}\frac{\sqrt{x'-iy'} \pm \sqrt{x'+iy'}}{\sqrt{x'^2+y'^2}}, \pm\frac{i}{2}\frac{\sqrt{x'+iy'} \mp \sqrt{x'-iy'}}{\sqrt{x'^2+y'^2}}\right)$

At the first three points above, $\frac{dy}{dx}$ has the values $\frac{y'}{x'-1}$, $\frac{2y' - \sqrt{3}}{2x'+1}$ and $\frac{2y' + \sqrt{3}}{2x'+1}$, respectively. These show that the tangents at these three points, which are the points representing the three cube roots of unity, pass through the pole (x',y'). This follows directly from the fact that (5), the first polar of (x',y') with respect to (4), passes through these three points. At the origin $\frac{dy}{dx} = \frac{x' \pm \sqrt{x'^2+y'^2}}{y'}$ i.e., the tangents to the curve at the origin are $y = \frac{x' \pm \sqrt{x'^2 + y'^2}}{y'}x$ which are orthogonal. If θ is the inclination of either of these tangents, $\tan 2\theta = -\frac{y'}{x'}$. Hence, <u>to construct the tangents to (4) at the origin, join the origin to the point $(x',-y')$ and bisect the angles made by this line with the x-axis. The bisectors are the required tangents.</u> These tangents form the only real degenerate conic of the pencil (2), and are obtained also by putting $\lambda = \infty$ in equation (2).

This is sufficient to enable us to make a fairly accurate drawing of the curve. (See figure, p. 9).

Some of the properties of (4) appear more readily if it is put into the polar form

(7) $\rho^2[\rho^3(x'\sin\theta - y'\cos\theta) - \rho\sin 3\theta + x'\sin 2\theta + y'\cos 2\theta] = 0.$

The factor ρ^2 indicates again, that the origin is a double

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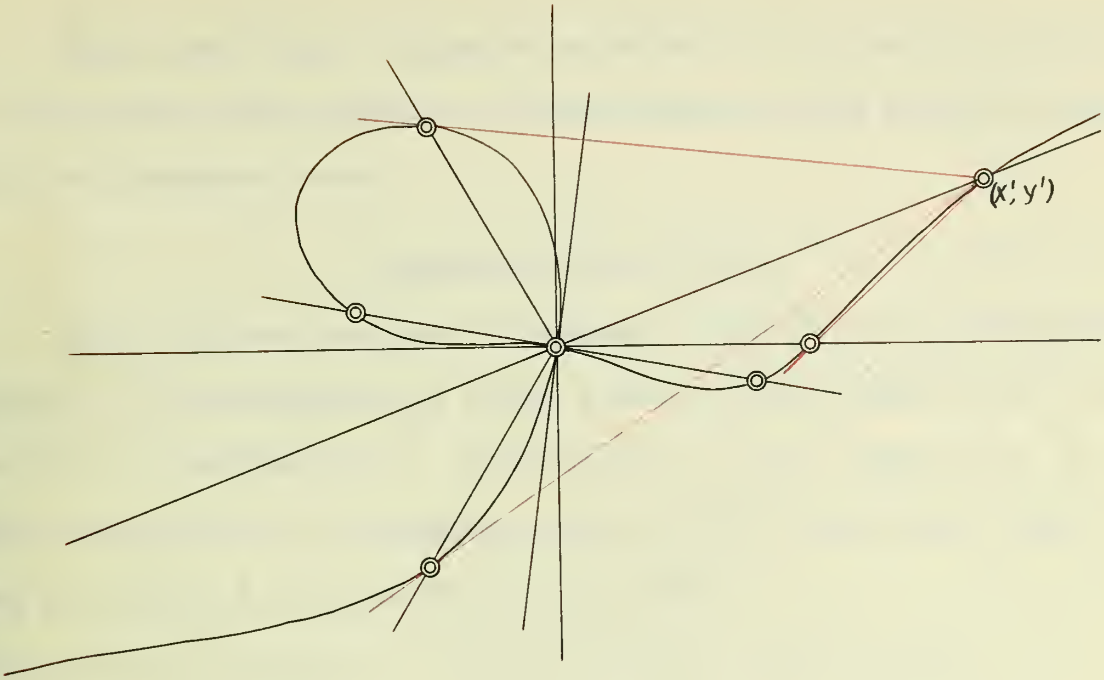

point. If $x'\sin 2\theta + y'\cos 2\theta = 0$, or $\tan 2\theta = -\frac{y'}{x'}$, one value of ρ is zero. The others are obtained from

$$\rho^2(x'\sin\theta - y'\cos\theta) = \sin 3\theta, \quad \text{whence}$$

$$\rho = \frac{1}{\pm\sqrt[4]{x'^2 + y'^2}},$$ provided we consider $\sin 2\theta$ negative and $\cos 2\theta$ positive. An interchange of signs would make ρ imaginary. (The fourth root arises from the fact that the functions of θ involve the square root.) The curve cuts one of the tangents at the origin in two points equidistant from the origin. These two points are the real base-points of (2) as may be verified by making use of the coordinates of the base-points as given on p. 8. Since the coefficient of ρ^2 within the bracket is zero, the sum of the three non-vanishing segments on any ray through the origin vanishes. The origin is therefore a center[1] of the curve.

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[1] See Kasner (l.c. p. 1), p. 397, for this definition of center.

More than this, the polar equation (7), gives us a hint as to the form of the equation of the product curve for $n > 2$. This will be discussed later.

Quadruple foci of (4).

Foci are sect-points of tangents from the circular points to a curve.[1] The tangents at I and J are of the form $y = ix + b$ and $y = ix + c$, respectively. Putting $b = \beta + i\alpha$, then (α, β) is the only real point on the tangent, i.e., it is the focus. Substituting $y = -ix + b$ in equation (4) we have

$$(8)\quad (4i+4ib^2x'+4b^2y')x^3 + (4ib^3y'-6b-2ix'-8b^3x'+2y')x^2 +(2bx'+2iby'-5ib^4x'-b^4y'-8ib^2)x+ b^3-b^2y' + b^5x' = 0$$

The degree reduces to 3 because the circular points are double points. In order for $y = -ix + b$ to be tangent to I, the coefficient of x^3 must also vanish, i.e.,

$$4i + 4ib^2x' + 4b^2y' = 0, \text{ or } b^2 = -\frac{1}{x'-iy'}.$$

Hence the tangents at I are $y = -i\left(x \pm \frac{1}{\sqrt{x'-iy'}}\right)$ Similarly, the tangents at J are $y = i\left(x \pm \frac{1}{\sqrt{x'+iy'}}\right)$. These intersect in the four points, (two of them real)

$$\left(\pm\frac{1}{2}\,\frac{\sqrt{x'-iy'} \pm \sqrt{x'+iy'}}{\sqrt{x'^2+y'^2}},\quad \frac{i}{2}\,\frac{\sqrt{x'+iy'} \mp \sqrt{x'-iy'}}{\sqrt{x'^2+y'^2}}\right)$$

which are the base-points of the pencil (2). We have seen that

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[1] Bassett, Elementary Treatise on Cubic and Quartic Curves, p. 46.
Charlotte A. Scott, Modern Analytical Geometry, p. 122.
Numerous special cases of foci are treated by R. A. Roberts, On Foci and Confocal Plane Curves, Quarterly Journal of Mathematics XXXV (1903-4) pp. 297-384.

the orthogonal tangents at the origin are the two lines of the real degenerate equilateral hyperbola of the pencil (2). Hence we may state the

Theorem III. <u>The three degenerate equilateral hyperbolas of the pencil (2) are the tangents to the curve (4) at its double points, which are their vertices. The base-points of the pencil (2) are foci of (4).</u>

<u>Single foci of (4).</u>

The quintic (4) has three double points and no other singularities. Its class is therefore $5(5 - 1) - 3 \cdot 2 = 14$. Since the circular points are double points we can draw from each of them only 10 tangents touching the curve elsewhere. The 100 intersections of these ten tangents are foci of the curve but only 10 of these are real. They belong to the 196 base-points of a pencil of curves of order 14. Each of the 10 tangents from I cuts each of the two tangents at J in two coincident points (double foci), thus yielding 40 double foci; similarly, the tangents from J determine 40 double foci. The tangents at I and J determine four quadruple foci (2 real) considered above, counting for 16 points. Thus we have accounted for $100 + 40 + 40 + 16 = 196$ base-points. To determine the real single foci, impose on equation (8) the condition that it shall have equal roots, i.e., that the discriminant shall vanish. To obtain the discriminant, take the derivative with respect to x and solve the quadratic so obtained for x. Since the double roots of the cubic (8) must also be roots of its derived

equation, we reverse the process and substitute the roots of the derived equation in (8). The two expressions thus obtained are the two factors of the discriminant. These, set equal to zero, are $b - ix' - y' = 0$, and

(9) $$4b^9(x'+iy')^3 + 27b^7(x'-iy')^2 - b^6(15ix'^3+117x'^2y'-45ix'y'^2 -39y'^3)+54b^5(x'-iy') - 54ib^4(x'-iy')^2 - 12b^3(x'+iy')^3 + 27b^3 - 27ib^2(x'-iy') - 4i(x'+iy')^3 = 0.$$

Hence, the line $y = -ix + ix' + y'$ is a tangent to the curve (4). The real point on it is (x',y'), the pole, which is therefore a focus. (As is well known, the corresponding value of c is $y' - ix'$) By equation (9), the other nine real single foci are so situated that the origin is their centroid and the product of their distances from the origin has an absolute value equal to unity. The former follows from the fact that the eighth degree term is missing; the latter is seen by dividing through by the coefficient of b^9, when the constant term reduces to i.

Since the inverse of a focus is a focus of the inverse curve,[1] the problem of finding the foci of (4) reduces to that of finding the foci of its inverse with respect to the origin, viz., a circular quartic

(10) $$(x^2+y^2)[2x'xy+y'(x^2-y^2)] + y^3 - 3x^2y + x'y - y'x = 0.$$

This does not simplify matters, however.

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[1] Bassett, (l.c. p. 10) p. 50.

Isotropic Coordinates.

The problem of finding foci is much simpler when the equation of the curve is expressed in isotropic coordinates.[1] Put $z = x + iy$, $\bar{z} = x - iy$, or $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$. Equation (4) becomes

$$(11)\qquad f \equiv [(x'-iy')\bar{z}^2 - 1]z^3 - [(x'+iy')(\bar{z}^3-1)]z^2 - \bar{z}^2(x'-iy') + \bar{z}^3 = 0.$$

$$(12)\qquad \frac{\partial f}{\partial z} = 3[(x'-iy')\bar{z}^2 - 1]z^2 - 2[(x'+iy')(\bar{z}^3-1)]z = 0.$$

The roots of (12) are $z = 0$, $z = \frac{2(x' + iy')(\bar{z}^3 - 1)}{3(x' - iy')\bar{z}^2 - 1}$.

Substituting $z = 0$ in (11), we get $\bar{z} = x' - iy'$, i.e., x', y' is a focus.[2] Substituting the second root of (12) in (11), we have

$$(13)\qquad 4(x'+iy')^3(\bar{z}^3-1)^3 + 27\bar{z}^2(-\bar{z}+x'-iy')[(x'-iy')\bar{z}^2-1]^2 = 0.$$

If in equation (13), z is replaced by its equivalent, $-ib$, the result is identical with equation (9), as it should be.

III. Transformation $z' = z - \frac{(n+1)(z^{n+1} - 1)}{(n+1)z^n} = \frac{1}{z^n}$

The pencil of stelloids connected with this transformation is the pencil of curves through the $n + 1^{th}$ roots of unity and their associates. The transformation represents a $(1,n)$ corres-

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[1] A. Perna. Le Equazioni delle Curve in Coordinate Complesse Coniugate, Rendiconti del Circolo Matematico di Palermo XVII (1903) pp. 65-72.
Beltrami, Ricerche sulla Geometria delle forme binarie cubiche, Memorie dell'Acc. di Bologna X (1870) p. 626.
Cesaro, Sur la determination des foyers des coniques, Nouvelles Annales des Mathematiques LX (1901) pp. 1-9.
G. Lery, Sur la fonction de Green, Annales Scientifiques de l'Ecole Normale Superieure, XXXII (1915) pp. 49-135.
C. E. Brooks, (l.c. p. 1) calls these conjugate coordinates. Cayley (Collected Works VI, p. 498) uses the name circular coordinates.

[2] Lery, (l.c.) p. 51. Brooks (l.c.) p. 309.

pondence between the pole (x',y') and the $\underline{n}$ real base-points of the first polar pencil of (x',y'). To establish the equation of the pencil and first polar in polar coordinates we have

$$u + iv = z^{n+1} - 1 = \rho^{n+1}\cos(n+1)\theta + i\rho^{n+1}\sin(n+1)\theta - 1 = 0.$$

The pencil of stelloids is

(14) $$u + \lambda v = \rho^{n+1}\cos(n+1)\theta - 1 + \lambda\rho^{n+1}\sin(n+1)\theta = 0.$$

The first polar pencil of (x',y') is

(15) $$u_1 + \lambda v_1 = \rho^n x'\cos n\theta - y'\rho^n\sin n\theta - 1 + \rho^n\lambda(x'\sin n\theta + y'\cos n\theta) = 0.$$

The product of (14 and (15) is

(16) $$\rho^n[\rho^{n+1}(x'\sin\theta - y'\cos\theta) - \rho\sin(n+1)\theta + x'\sin n\theta + y'\cos n\theta] = 0,$$

which may be written in the form

(17) $$\rho^{2n}(x'y - y'x) - \rho^{n+1}\sin(n+1)\theta + x'\rho^n\sin n\theta + y'\rho^n\cos n\theta = 0.$$

This shows that in cartesian coordinates, $(x^2 + y^2)^n$ is a factor of the terms containing x and y to degree 2n + 1 while the next highest power of x and y is n + 1. Hence the

Theorem IV: <u>The product of the pencil of stelloids determined by the n + 1th roots of unity and their associates as base-points and the first polar pencil of a point (x',y'), is a circular curve (16) having an n-fold point at each of the circular points and at the origin.</u>

Also since $x'y - y'x$ is a factor of the highest degree terms, and the terms of order n + 2 to 2n are missing, we have

Theorem V: <u>The line $x'y - y'x = 0$, joining the origin and the pole is an asymptote of the product (16), if $n > 1$. The sum of segments on rays thru the origin is zero, i.e., the origin is a center.</u>

If in the second factor of (16) we put $\rho = 0$, we get $x'\sin n\theta + y'\cos n\theta = 0$, that is

Theorem VI: The tangents to (16) at the origin are the lines $y = x \tan\theta$, where $\tan n\theta = -\frac{y'}{x'}$. These n tangents divide the whole angle about the origin into n equal parts, beginning at $y = x\tan\varphi$, where $\varphi = \arctan(-\frac{y'}{x'}) \div n\theta + 2k\pi$.

Making use of the values $\theta = \frac{\varphi}{n} - \frac{2k\pi}{n}$, $(k = 0,1,2,----(n-1))$ we find that the curve cuts the tangents at the origin in the points $\rho^n(x'\sin\theta - y'\cos\theta) = \sin(n+1)\theta$, or $\rho^n = \pm\frac{1}{\sqrt{x'^2+y'^2}}$, according as $\cos n\theta$ or $\sin n\theta$ is considered positive, i.e., the curve cuts in other real points all of these tangents if n is odd and cuts only half of them elsewhere if n is even. The points of intersection are the base-points of (15), as may be easily verified by substituting their coordinates in (15).

The tangents at the origin constitute the degenerate curve obtained by making $\lambda = \infty$ in (15). A general theorem[1] states that if two corresponding curves C^m and C^n in two projective pencils of curves have a common multiple point of multiplicities r and $s (r < s)$ respectively, their product K has there a multiple point of order r and the r tangents of K are tangents to C^m. We have here an example in which both C^m and C^n are the real degenerate members of the two pencils. In fact, each of them consists of straight lines through the origin, the C^n being the $n + 1$ straight lines through the origin obtained by making $\lambda = \infty$ in (14).

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[1] Sturm, l.c. Ency. der Math. Wiss. III 2,3, p. 355.

Quadruple Foci of (16).

In rectangular coordinates, equation (16) is

$$(18)\quad (x^2+y^2)^n(x'y-y'x)-(n+1)y[x^n+\frac{i^2n(n-1)}{3!}x^{n-2}y^2+\cdots\cdots + \frac{i^{2(r-1)}n(n-1)\cdots(n-2r+3)}{(2r-1)!}x^{n-2r+2}y^{2r-2}+\cdots] = 0.$$

The terms in the bracket are the even terms of the binomial expansion $(x + iy)^{n+1}$.

Any line $y = ix + b$ thru the circular point I cuts the curve (18) n times at I and in $n + 1$ other points. To make the line tangent at I, impose the condition that it shall have $(n + 1)$ intersections at I, i.e., the coefficient of the highest power of x must vanish when $y = ix + b$ is substituted in (18). Making the substitution and picking out the coefficient of x^{n+1}, we have the condition

$$(19)\quad 2^nb^ni^{n+1}x'-2^nb^ni^ny'-i[n+1+\frac{i^4(n+1)n(n-1)}{3!}+\cdots + \frac{i^{4(r-1)}(n+1)n(n-1)\cdots(n-2r+3)}{(2r-1)!}] = 0.$$

Since in the bracket i appears only to multiples of the fourth power, we have simply the sum of the coefficients of the even terms in the binomial expansion $(x + y)^{n+1}$. This sum[1] is 2^n. Hence, equation (19) becomes

$$(20)\quad 2^nb^ni^{n+1}x' - 2^nb^ni^ny' - 2^ni = 0,\ \text{or}\ b^n = \frac{1}{i^n(x'+iy')}.$$

Setting $x' + iy' = re^{i\theta}$, $b^n = re^{i(-\theta-n\pi+2k\pi)}$, the n values of b are $b_k = \sqrt[n]{r}\cdot e^{i(\frac{\theta}{n}+\frac{2k\pi}{n})}$, $k = (0,1,2,\ \text{------}(n - 1))$

The tangents at I are therefore

$$(21)\quad y = ix + b_k = i(x - \frac{1}{\sqrt[n]{x'+iy'}})$$

[1] Hagen, Synopsis uer Höheren Mathematatik, p. 64.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

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The form of b_k shows that in the complex plane the values of b represent n equidistant points on a circle of radius $\frac{1}{\sqrt[n]{r}}$. This is analogous to the location of tangents at the origin which divide the whole angle about the origin into n equal parts beginning at the asymptote $x'y - y'x = 0$.

The tangents at J are obtained from (21) by changing the sign of <u>i</u>, since a tangent through J may be written in the form $y = -ix+b$. These are

(22) $$y = -ix + \bar{b}_k = -i\left(x - \frac{1}{\sqrt[n]{x'-iy'}}\right)$$

Conjugate tangents (21) and (22) intersect in the real points (for the same value of k)

$$(x,y) \equiv \left[\frac{\sqrt[n]{x'+iy'} + \sqrt[n]{x'-iy'}}{2\sqrt[n]{x'^2+y'^2}},\ i\,\frac{\sqrt[n]{x'+iy'} - \sqrt[n]{x'-iy'}}{2\sqrt[n]{x'^2+y'^2}}\right]$$

By substitution in (15), these may easily be shown to be the base-points of the first polar pencil (15). Hence the

Theorem VII: <u>The tangents to (18) at the circular points (and at the origin) pass through the base-points of the first polar pencil (15).</u>

Putting $\lambda = i$ in equation (15) we get the degenerate form $\rho^n(\cos n\theta + i \sin n\theta) = \frac{1}{x'+iy'}$, which reduces to $\rho(\cos\theta + i \sin\theta) = \frac{1}{\sqrt[n]{x'+iy'}} = x+iy$, or $y = ix + \frac{1}{\sqrt[n]{x'+iy'}}$ which is identical with the equation (21). Similarly, putting $\lambda = -i$, we get (22). This, in connection with the fact that the tangents at the origin constitute a degenerate curve of the pencil (15), gives us

Theorem VIII: <u>Three degenerate curves of the pencil (15) which break up into n straight lines have their singular points at the origin and the circular points I and J, respectively, each of</u>

which is an n-fold point of the curve (16), and these 3n lines are tangents to the curve at the n-fold points.

Single foci of (16).

Introducing isotropic coordinates $z = x + iy = \rho(\cos\theta + i\sin\theta)$; $\bar{z} = x - iy = \rho(\cos\theta - i\sin\theta)$ there is $z\bar{z} = \rho^2$;

$$z^{n+1} = \rho^{n+1}[\cos(n+1)\theta + i\sin(n+1)\theta]; \quad \bar{z}^{n+1} = \rho^{n+1}[\cos(n+1)\theta - i\sin(n+1)\theta];$$

$$\rho\cos\theta = \frac{z+\bar{z}}{2}; \quad \rho\sin\theta = \frac{z-\bar{z}}{2i}; \quad \rho^{n+1}\sin(n+1)\theta = \frac{z^{n+1}-\bar{z}^{n+1}}{2i};$$

$$\rho^n \sin n\theta = \frac{z^n-\bar{z}^n}{2i}; \quad \rho^n\cos n\theta = \frac{z^n+\bar{z}^n}{2}.$$

Substituting these values in equation (16), it reduces to

$$(23)\quad f \equiv [(x'-iy')\bar{z}^n-1]z^{n+1}-[(x'+iy')(\bar{z}^{n+1}-1)]z^n+\bar{z}^{n+1}-(x'-iy')\bar{z}^n=0.$$

To find the foci, impose the condition on (23) that it shall have equal roots in z.[1] To do this, we get

$$(24)\quad \frac{\partial f}{\partial z} = [(x'-iy')\bar{z}^n-1](n+1)z^n-n[(x'+iy')(\bar{z}^{n+1}-1]z^{n-1}= 0.$$

If a root of (24) is also a root of (23), it is a double root of (23). Equation (24) has (n - 1) roots $z = 0$. In order for $z = 0$ to be a root of (23), we must have $\bar{z}^n[\bar{z}-(x'-iy')] = 0$, i.e., $\bar{z} = x' - iy'$, whence the pole (x',y') is a focus. $\bar{z} = 0$ signifies merely that the origin is a multiple point. The remaining root of (24) is $z = \frac{n[(x'+iy')(\bar{z}^{n+1}-1)]}{(n+1)[(x'-iy')\bar{z}^n-1]}$. To find the condition that this shall be a root of (23) it is substituted in (23) giving the condition

$$(25)\quad n^n[(x'+iy')(\bar{z}^{n+1}-1)]^{n+1}-\bar{z}^n(n+1)^{n+1}[\bar{z}-x'+iy'][(x'-iy')\bar{z}^n-1]^n=0.$$

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[1] See Lery or Brooks, l.c. p. 14.

The highest power of $\bar{z}$ in this equation is $(n+1)^2$ and the next highest power is $n^2 + n + 1 = (n+1)^2 - n$. Hence, for $n > 1$, the coefficient of the next highest power of $\bar{z}$ vanishes and the origin is the centroid of the roots of (25), i.e., of the single foci of (23), or (16). Also the constant term of (25) arises in the first bracket and has the same coefficient, except for sign, as the highest power of $\bar{z}$, i.e., the product of the roots of (25) is ± 1, according as n is even or odd.

We note in passing that if in (23) we set the coefficient of z^{n+1} equal to zero, we get at once the double foci obtained in the preceding section.

<u>First polar of (16)</u>.

By theorem I, the product of (14) and the second polar pencil of (x',y') is the first polar of (16), viz.,

$$(26) \quad \rho^{n-1}\{\rho^{n+1}[(x'^2-y'^2)\sin 2\theta - 2x'y'\cos 2\theta] - \rho^2\sin(n+1)\theta + (x'^2-y'^2)\sin(n-1)\theta + 2x'y'\cos(n-1)\theta\} = 0.$$

Since the difference in degree of the two highest powers of ρ is $n - 1$, for $n > 2$, the asymptotes are determined by

$$(27) \qquad (x'^2-y'^2)\sin 2\theta - 2x'y'\cos 2\theta = 0.$$

From this, $\tan 2\theta = \frac{2m}{1 - m^2}$, where $m = \frac{y'}{x'}$. Moreover, since $\tan 2\theta = \tan 2(\theta + \frac{\pi}{2})$, it follows that these are the lines joining (x',y') to the origin and the line normal to it at the origin.

The tangents at the origin are determined by

$$(28) \quad (x'^2-y'^2)\sin[(n-1)\theta + 2k\pi] + 2x'y'\cos[(n-1)\theta + 2k\pi] = 0.$$

since in (26) this is the condition for a root $\rho = 0$. From (28)

we get $\tan(n-1)\theta = -\frac{2x'y'}{x'^2-y'^2} = \tan 2[\arctan(-\frac{y'}{x'})]$,

or $(n-1)\theta = 2\varphi + m\pi = -2A + m\pi$, where A is the inclination of the line joining the origin and the pole (x',y'). For $\sin(n+1)\theta = 0$, $\rho^{n+1} = \cos(n+1)\theta$, or $\rho = 1$. Hence the curve (26) passes thru the $(n+1)^{th}$ roots of unity. But the curve (16) with respect to which (26) is the first polar of (x',y'), also passes through these points. We have therefore the

Theorem IX: The lines joining the pole (x',y') to the $(n+1)^{th}$ roots of unity are tangents to the curve (16).

IV. The general transformation $z' = z - \frac{(n+1)f(z)}{f'(z)}$.

In the general case[1] $f(z) = u + iv = a_0\prod_1^{n+1}(z-z_i) = 0$. This may be thought of as representing $n+1$ lines[2] through the circular point I. The pencil of stelloids is $u + \lambda v = 0$ and the $n+1$ lines are determined by the value $\lambda = i$.

Similarly, for $\lambda = i$, the first polar pencil $u_1 + \lambda v_1 = 0$ represents n lines through I and the base-points of the first polar pencil. Also $u - iv = 0$ and $u_1 - iv_1 = 0$ represent sets of lines through J.

By the general theorem regarding multiplicities of products,

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[1] A. Emch (l.c. p. 2.); p. 2.

[2] Compare C. Segre. Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici. Math. Annalen XL (1892) pp.413-467.

p. 15, we then have (assuming that the general theorem applies to imaginary elements)

Theorem X: The product $uv_1 - u_1v = 0$ of the projective pencils $u + \lambda v$ and $u_1 + \lambda v_1$ has an n-fold point at each of the circular points and the n lines $u_1 + iv_1 = 0$ are tangents to the product at I and the n lines $u_1 - iv_1$ are tangents to J.

Since the sect-points of $u_1 + iv_1 = 0$ and $u_1 - iv_1 = 0$ are the base-points of the pencil $u_1 + \lambda v_1 = 0$, and these lines are tangents at I and J, we have

Theorem XI: The n^2 base-points of the first polar pencil $u_1 + \lambda v_1 = 0$ are quadruple foci of the product $uv_1 - u_1 v = 0$. Among these are the n real base-points forming n real foci.

Since in the special cases treated, the pole is a focus, we might expect that the pole is also, in general, a focus. This, however, is not the case.

Equation 27, p. 10 of the article by Emch referred to above, is the equation of the product in general, viz.,

(29) $(x - x')(rv - su) - (y - y')(ru + sv) = 0$, where *r* and *s* are $\frac{u_x'}{n+1}$ and $\frac{v_x'}{n+1}$ respectively. The form of (29) gives us the

Theorem XII: The product curve is also the product of the pencil of lines $(x - x') - \lambda(y - y') = 0$, through the pole, and the pencil of circular curves $(ru + sv) - \lambda(rv - su) = 0$.

The line $x + iy = x' + iy'$, joining the pole and the circular point I, meets this curve in points of $(ru + sv) + i(rv - su) = 0$, (30)

which is identical with the expression just above equation (4), p.4

[illegible]

of that article, where it is shown to be equal to

(31) $\quad (a_0+ib_0)\prod_{1}^{n+1}(x+iy-z_i)\prod_{1}^{n}(x-iy-\bar{z}_k) = 0.$

Substituting the value of $x = x' + iy' - iy$, the first two sets of factors become constants and the third one gives n values of y which are

(32) $\quad y = \dfrac{x'+iy'-\bar{z}_k}{2i}, \qquad (k=1,2,\cdots,n).$

Hence two values of y cannot be equal unless two points of $\bar{z}_k$ coincide. In the special cases treated, $\bar{z}_k = 0$ so that the pole is a focus, but in the general case, we have

Theorem XIII: <u>In general the pole (x',y') is not a focus of the product curve.</u>

This also follows directly, since (30) is independent of (x',y').

From the equation of the tangent line and equation (32) the point of tangency is *(the contact is of order n-1)*

(33) $\quad \left(\dfrac{x'+iy'}{2}, \dfrac{x'+iy'}{2i}\right)$

for the case $\bar{z}_k = 0$. This point lies on the line $y = -ix$. In the same way it may be shown that the point of contact of the tangent joining the pole to the circular point J lies on the line $y = ix$. Hence the

Theorem XIV: <u>In the special cases of sections II and III, the circular points, the pole, the origin, and the two points of contact of the tangents joining the pole to the circular points, are the vertices of a complete quadrilateral, i.e., the points of contact of these tangents are the associate points of the pole and origin.</u>

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ACKNOWLEDGMENT.

I am under great obligations to Professor Emch for assistance in the preparation of this thesis. To him and to Professor Townsend I am very much indebted for continued encouragement and assistance during my work as a graduate student in the University of Illinois.